Level Splitting in Association with the Multiphoton Bloch-Siegert Shift

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Abstract. We present a unitary equivalent spin-boson Hamiltonian in which terms can be identified which contribute to the Bloch-Siegert shift, and to the level splittings at the anticrossings associated with the Bloch-Siegert resonances. First-order degenerate perturbation theory is used to develop approximate results in the case of moderate coupling for the level splitting.

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1. Introduction

The dynamics of a two-level with sinusoidal coupling has been of interest since the time of Bloch and Siegert [1, 2]. The (closely related) basic model for a two-level system coupled to a simple harmonic oscillator was considered by Cohen-Tannoudji et al [3]. The coupling in these models produces an increase in the two-level system transition energy (sometimes termed the Bloch-Siegert shift). As the coupling strength is increased, the levels shift relative to one another, producing both level crossings and level anticrossings. Level crossings occur when the dressed two-level transition energy matches an even number of oscillator quanta (in which case the parity of the states are mismatched, so no mixing occurs). Level anticrossings occur when the dressed two-level transition energy is resonant with an odd number of oscillator quanta, with the magnitude of the splitting indicative of the ability of the coupled system to convert energy between the two different degrees of freedom.

These models were studied initially in the context of spin dynamics in a magnetic field [1, 2], but they also appear in other applications. The coupling between atoms and an electromagnetic field can in some cases be described by these models, in which case the resonances mentioned above correspond to multiphoton interactions. Multiphoton resonances in which a substantial number of photons are exchanged have become experimentally accessible recently [4]. In part because of this there has been renewed interest in the multiphoton regime [5, 6].

We have found a unitary transformation which produces a rotated version of the problem which appears to provide a clean separation between terms which produce most of the Bloch-Siegert shift, and terms which produce the level splitting at the anticrossings. This is interesting because it allows us to develop estimates when the coupling is moderately strong for both the shift and the splittings using conventional methods on the rotated problem. In essence, we are able to capture most of the level splitting in the multiphoton regime in terms of first-order coupling in the context of degenerate perturbation theory. This provides a new way to look at the problem which may be useful.

2. Unitary equivalent Hamiltonian

The Hamiltonian for the coupled two-level system and oscillator of interest (the spin-boson Hamiltonian) can be written as

$$\hat{H} \;=\; \frac{\Delta E}{2}\hat{\sigma}_z + \hbar\omega_0 \hat{a}^\dagger \hat{a} + U(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x \tag{1}$$

where the $\hat{\sigma}_i$ are the Pauli matrices. Since we are interested in the multiphoton regime, we assume that the background excitation of the photon field is large:

$$\Delta E \gg \hbar\omega_0 \;, \quad n \gg 1$$

Rotations are often used to simplify Hamiltonians [7]; however, in this case our rotation will make the problem more complicated mathematically but perhaps simpler functionally as outlined above. We consider the unitary equivalent Hamiltonian

$$\hat{H}' \;=\; \hat{\mathcal{U}}^\dagger \hat{H} \hat{\mathcal{U}} \tag{2}$$

where

$$\hat{\mathcal{U}} \;=\; \exp\left\{ -\frac{i}{2} \arctan\left[\frac{2U(\hat{a}+\hat{a}^\dagger)}{\Delta E}\right] \hat{\sigma}_y \right\}$$

The rotated Hamiltonian, $\hat{H}'$ can be broken up into an unperturbed part ($\hat{H}_0$), and pieces which will be considered as perturbations ($\hat{V}$ and $\hat{W}$):

$$\hat{H}' \;=\; \hat{H}_0 \;+\; \hat{V} \;+\; \hat{W} \tag{3}$$

where

$$\hat{H}_0 \;=\; \sqrt{\Delta E^2 + 4U^2(\hat{a}+\hat{a}^\dagger)^2}\,\frac{\hat{\sigma}_z}{2} \;+\; \hbar\omega_0 \hat{a}^\dagger \hat{a} \tag{4}$$

$$\hat{V} = \frac{i\hbar\omega_0}{2} \left\{ \left[\frac{\dfrac{U}{\Delta E}}{1 + \left[\dfrac{2U(\hat{a}+\hat{a}^\dagger)}{\Delta E}\right]^2} \right] (\hat{a} - \hat{a}^\dagger) \right.$$
$$\left. +\; (\hat{a} - \hat{a}^\dagger) \left[\frac{\dfrac{U}{\Delta E}}{1 + \left[\dfrac{2U(\hat{a}+\hat{a}^\dagger)}{\Delta E}\right]^2} \right] \right\} \hat{\sigma}_y \tag{5}$$

$$\hat{W} = \hbar\omega_0 \left\{ \frac{\dfrac{U}{\Delta E}}{1 + \left[\dfrac{2U(\hat{a}+\hat{a}^\dagger)}{\Delta E}\right]^2} \right\}^2 \tag{6}$$

In the multiphoton regime of interest here ($n \gg 1$ and $\Delta E \gg \hbar\omega_0$), the last term produced by the rotation, $\hat{W}$, is small; we will therefore neglect it in what follows.

3. Eigenvalues of $\hat{H}_0$

Consider first the eigenvalue equation of the unperturbed Hamiltonian $\hat{H}_0$ in the rotated frame

$$E\psi \; = \; \hat{H}_0\psi \tag{7}$$

Separation of variables allows us to develop solutions of the form

$$\psi \; = \; u \otimes |s, m\rangle \tag{8}$$

where u satisfies

$$\left(E + \frac{\hbar\omega_0}{2}\right) u(y) \; = \; \frac{\hbar\omega_0}{2}\left[-\frac{d^2}{dy^2} + y^2\right] u(y) + m\sqrt{\Delta E^2 + 8V^2y^2}u(y) \tag{9}$$

Both from numerical calculations and the WKB approximation we have found that the energy eigenvalues are given approximately by

$$E_{n,m}(g) \; = \; \Delta E(g)m + \hbar\omega_0 n \tag{10}$$

The WKB approximation can be used to develop a useful analytic approximation to the dressed two-level transition energy $\Delta E(g)$, which we may write as

$$\Delta E(g) \; = \; \frac{\Delta E}{\pi}\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \sqrt{\frac{1 + 8g^2y^2/n}{\epsilon - y^2}} dy \tag{11}$$

with $\epsilon = 2n + 1$. The dimensionless coupling constant g is

$$g \; = \; \frac{U\sqrt{n}}{\Delta E} \tag{12}$$

In the limit of large n and $\hbar\omega_0 \ll \Delta E$ where this is valid, the rotated system governed by $\hat{H}_0$ alone behaves like a dressed two-level system (with increased transition energy) and an unperturbed oscillator.

The condition for Bloch-Siegert resonances can be written as

$$\Delta E(g) \; = \; (2k + 1)\hbar\omega_0 \tag{13}$$

Level crossings occur in the modified version of the problem described by the unperturbed Hamiltonian $\hat{H}_0$ in the rotated frame. As level anticrossing occur in the original spin-boson model at these resonances, the coupling that is responsible for the level anticrossing has been eliminated in $\hat{H}_0$. This is an interesting and perhaps unexpected feature of this rotation.

4. Level splitting in the unrotated Hamiltonian

Near a resonance, we can use a two-level description to account for the level splittings.

$$E(g)\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} E_0(g) & v \\ v & E_1(g) \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \tag{14}$$

Two levels with energies E_0 and E_1 that depend on the dimensionless coupling strength g cross, and couple to each other with an interaction v which we assume to be constant in the vicinity of the resonance. At resonance (g_0), the two levels in this simplified model are degenerate

$$E_0(g_0) = E_1(g_0) \tag{15}$$

The splitting between the two levels at this point is twice the magnitude of the interaction

$$\Delta E_{min} = E_+(g_0) - E_-(g_0) = 2|v| \tag{16}$$

The level splittings in the case of weak coupling have been known for some time [2], as mentioned above. Shirley's results written in our notation are

$$\Delta E_{min} = \frac{g_0^{2k+1}}{2^{2k-1}(k!)^2}\left(\frac{\Delta E}{\hbar\omega_0}\right)^{2k}\Delta E \tag{17}$$

We have plotted results from the direct numerical solution of the spin-boson Hamiltonian [Equation (1)], and also for the this weak coupling result in figure 1. When the dimensionless coupling constant g is small the results match well; when the coupling gets stronger, we see (as expected) that perturbation theory begins to break down.

5. Level splitting in the rotated Hamiltonian

The dressed transition energy of the two-level system is described reasonably well through the unperturbed part $\hat{H}_0$ of the rotated Hamiltonian, but no level splittings occur in the eigenvalues of $\hat{H}_0$. Hence, all of the splitting must be due to the terms we have considered to be perturbations. In this section, our goal is to apply degenerate perturbation theory in the rotated frame to see whether the larger of the perturbation terms $\hat{V}$ can account for the level splitting.

To calculate the level splitting in the vicinity of an anticrossing, we need to compute the eigenkets $\psi_{n,m}$ of $\hat{H}_0$ where

$$\hat{H}_0\psi_{n,m} = E_{n,m}\psi_{n,m} \tag{18}$$

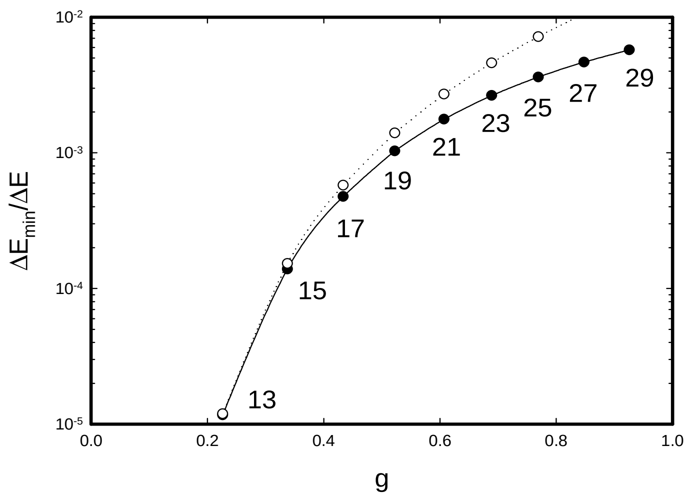

Figure 1. Energy level splitting on resonance as a function of the dimensionless coupling strength for $\Delta E = 11\hbar\omega_0$ at large n. Exact numerical results: full circles; literature results: open circles. The odd integers label the Bloch-Siegert resonance $2k+1$.

This can be done numerically, or by using the WKB approximation (which we have found to be effective for such problems). Near the $(2k+1)$th resonance, the level anticrossing is well-described by a simple two-level approximation

$$E(g)\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} E_{n,m}(g) & \langle \psi_{n,m}|\hat{V}|\psi_{n+2k+1,m-1}\rangle \\ \langle \psi_{n+2k+1,m-1}|\hat{V}|\psi_{n,m}\rangle & E_{n+2k+1,m-1}(g) \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \quad (19)$$

The energy splitting at resonance is

$$\Delta E_{min} = E_+(g_0) - E_-(g_0) = 2|\langle \psi_{n,m}|\hat{V}|\psi_{n+2k+1,m-1}\rangle| \quad (20)$$

In figure 2 we have plotted level splittings taken from a direct numerical solution of the original spin-boson Hamiltonian [Equation (1)] and also from first-order degenerated perturbation theory as discussed here (we used numerical solutions for the eigenfunctions $\psi_{n,m}$ for this result). We can see from figure 2 that the exact numerical results for the level splitting of the unrotated Hamiltonian match very well the results obtained by using degenerate perturbation theory on the rotated Hamiltonian. Minor deviations occur at the larger g values which we attribute to the omission of higher-order terms in the degenerate perturbation theory.

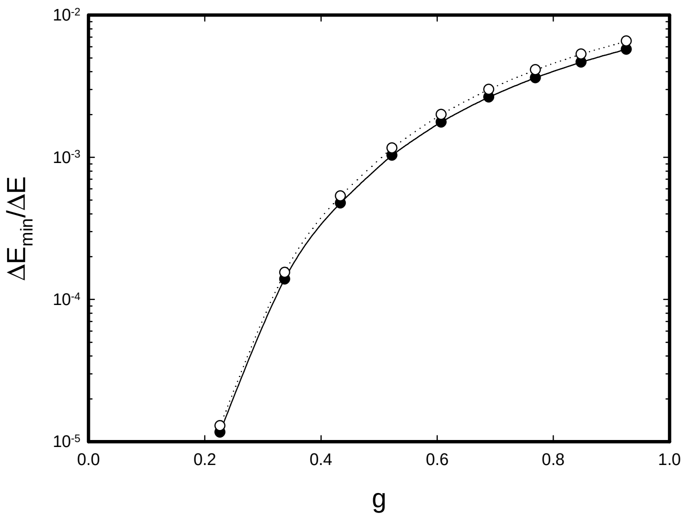

Figure 2. Energy level splitting on resonance as a function of the dimensionless coupling strength for $\Delta E = 11\hbar\omega_0$ at large n. Exact numerical results: full circles; first-order splitting from (20): open circles.

6. Conclusion

We have found a useful unitary transformation that produces a rotated Hamiltonian for the spin-boson problem in the multiphoton regime that has interesting properties. The rotated Hamiltonian is more complicated mathematically than the initial spin-boson Hamiltonian, but appears to be simpler in terms of functionality. One part of the rotated Hamiltonian is identified as an unperturbed Hamiltonian ($\hat{H}_0$) which appears to describe the coupled systems reasonably well away from the level anticrossings. This part of the problem is useful for developing estimates of the Bloch-Siegert shift. Another part of the rotated Hamiltonian ($\hat{V}$) is identified as a perturbation which is responsible for most of the coupling which occurs at the anti-crossing. Used with first-order degenerate perturbation theory, this term provides a reasonable approximation for the level splittings at the Bloch-Siegert resonances. Finally, there is present an additional term ($\hat{W}$) in the rotated Hamiltonian which is small (so that we have neglected it in our discussion here), but which can provide a minor correction to the dressed two-level system energies.

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